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Transonic equivalence rule: a nonlinear problem involving lift

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The inviscid transonic flow past a thin wing having swept leading edges, with smooth lift and thickness distributions, is shown to possess an outer nonlinear structure determined principally by a line source and a line doublet. Three domains (the thickness-dominated, the intermediate, and the lift-dominated), representing different degrees of lift control of the outer flow, are identified; a transonic equivalence rule valid in all three domains is established. Except in one domain, departure from the Whitcomb–Oswatitsch area rule is significant; the equivalent body corresponding to the source effect has an increased cross-sectional area depending nonlinearly on the lift. This nonlinear lift contribution results from the second-order corrections to the inner (Jones) solution, but produces effects of first-order importance in the outer flow. Of interest is an after-body effect dependent on the vortex drag, which is not accounted for by the classical transonic small-disturbance theory.

1. Introduction

The transonic flows of importance in aeronautics are mostly three-dimensional. Using modern computers, the problems are amenable to numerical analysis, which is however costly and difficult to perform. (See Nieuwland & Spee 1973; Hedman & Berndt 1973; Bauer, Garabedian, Korn & Jameson 1974.) This paper concerns one special, but rather important, class of three-dimensional transonic flows, to which an equivalence rule is applicable.

The transonic area rule or the equivalence rule of Whitcomb (1952, 1956) and Oswatitsch (1952) may be stated as follows. At transonic speed, the *outer* flow far from the body is the same as that produced by an (equivalent) body of revolution with the same axial distribution of cross-sectional area.† The results of Oswatitsch & Keune (1954), based on the slender-body formulation, were improved in formalism (Cole & Messiter 1957; Messiter 1957) and in scope (Ashley & Landahl 1965). Among the cited works, the most complete is that of Ashley & Landahl (1965), in which the restriction to an incidence small compared with body thickness is removed. (Also see Guderley 1962; Ferrari & Tricomi

† The counterparts of this area rule in the linear theories are not considered here. Refer to Oswatitsch's (1957) review for the subtle connexions among all 'area rules'. The equivalence rule may also be applied to the drag problem; but Berndt (1956) pointed out that, in the case of a straight trailing edge, it is applicable to drag correlation only if the tail sections are cylindrical. Cf. § 5.1 below.

1968.) An extension of the formulation to wings of unit-order aspect ratio was made by Spreiter & Stahara (1971), who considered a triangular wing carrying a body with lift.

Implicit in the cited works is the assumption of a flow model consisting of *two* distinct regions: an inner region governed by a linear transonic equation, the same as that in the slender-body theory (Munk 1924; Jones 1946; Ward 1949), and an outer nonlinear region which is *axisymmetric*. The axisymmetric assumption may be justified on the basis of the multipole expansion for the far field of an irrotational flow, applied to the cross-flow plane of a slender body. (See e.g. Batchelor 1967, pp. 117–124.) Accordingly, the effect produced by the line doublet and higher multipoles decays with distance from the axis much more rapidly than that of a line source. The outer flow may thus be expected to be dominated by an equivalent line source, and remains nearly axisymmetric. It should have been obvious that the stated rule cannot hold for lifting wings with an arbitrarily small thickness, as first noted by Hayes (1954). However, Hayes's formulation is restricted to slender wings with zero camber, and is valid only if thickness dominates. (See § 2.2.)

An equivalence rule involving lift based on a line source and a line doublet has been given in earlier work of ours (Cheng & Hafez 1972*a*, *b*). Barnwell (1973) revealed quite clearly an error in our work, which results from omission of the source term associated with a nonlinear correction in the matching; this would lead to different scaling laws for the outer flow in the lift-dominated case. But Barnwell (1973) precluded, among several important contributions, the logarithmic dependence on the expansion parameter for the inner solution. (See § 2.4). The outer solution obtained cannot satisfy the condition prescribed at the far boundary. Although far from being correct, Barnwell's analysis does contain one significant element. Namely, there is a nonlinear lift contribution to the equivalent source. In the present paper, the problem is given a thorough and systematic treatment, with regard to the matching and scaling in particular. The most significant theoretical point of this new development is perhaps the uncovering of the contribution of *several* second-order nonlinear corrections to the inner solution (not completely accounted for by the classical transonic small-disturbance theory), which produce a far-reaching influence in the outer flow, unsuspected in all previous work (to the best of our knowledge).

The material presented here is drawn largely from unpublished work (Cheng & Hafez 1973*a*); an outline of the analysis has been sketched in a greatly condensed note (Cheng & Hafez 1973*b*). This paper presents the full theory, adding several important interpretations and simplifications. The formulation of the inner problem has also been partly rearranged and expanded to improve clarity.

The local strengths of line doublet and line source defining the outer flow are, respectively, $F(x)/\rho_\infty U$ and $UdS_e(x)/dx$. $F(x)$ is the local lift force (integrated over the wing upstream of the x station); and $S_e(x)$ is the cross-sectional area of an equivalent body corresponding to the line source, also dependent on the lift distribution. The principal result of the rather lengthy analysis to follow can be compressed into a single equation. This expresses the nonlinear lift contribution to the equivalent area $S_e(x) - S_c(x)$ in terms of dF/dx and two double integrals involving streamwise and cross-stream velocity jumps $[u]$ and $[v]$,

$$\left.\begin{aligned} E(x) &= \frac{\rho_\infty}{4\pi}\int_{-b}^{b}\int_{-b}^{b}[v(x,y)]\,[v(x,s)]\log\left|\frac{b}{y-s}\right|dy\,ds, \\ T(x) &= \frac{\rho_\infty}{4\pi}\int_{-b}^{b}\int_{-b}^{b}[u(x,y)]\,[u(x,s)]\log\left|\frac{b}{y-s}\right|dy\,ds. \end{aligned}\right\}$$

$E(x)$ may be identified with the cross-flow kinetic energy, comparable to Prandtl's induced drag. The result in question is

$$S_e(x) - S_c(x) = \frac{1}{\rho_\infty a_\infty^2}\left\{\frac{r+1}{2}\left[(8\pi\rho_\infty U^2)^{-1}(2\,|\log\epsilon| + 1)\,F'(x)^2 + T(x)\right] + E(x)\right\}.$$

ϵ is the ratio of the inner to the outer lateral scale. This may be compared with its dimensionless form arrived at in (5.3).

The novel features of the nonlinear corrections uncovered here could have been anticipated from elementary gas dynamics. For, any departure from the sonic condition should result in an increase in the stream tube area from that at the throat, hence in the cross-sectional area of the equivalent body. But this fails to explain the increase in regions where the flows are not *exactly* sonic. The results may, nevertheless, be understood in terms familiar in fluid mechanics. Thus, additional space in § 2 is devoted to a delineation of such effects, where the basic assumptions of the present work and the strategy of the expansion procedure are also explained. The full basis of the theory is presented in § 3, where the inner solution and its non-uniformity are treated. The outer problem is formulated in § 4, where matching establishes the equivalence rule. The consequences and implications of the theory are discussed in § 5.

The departure from the classical area rule will generally increase with increasing lift and aspect ratio, and will also increase with decreasing thickness and leading-edge sweep. In the design range of transonic transport aircraft, the need for compensating the lift effect proves to be significant indeed (Cheng & Hafez 1973*a*, table 1). This paper will not treat aspects of design applications which belong to a separate study.

2. Assumptions and general remarks

2.1. *The inviscid model and equations*

A steady inviscid flow with a uniform free stream is considered. The slightly perturbed flow in this case can be described by a velocity potential corresponding to a uniform entropy, subject to a relative error proportional to the square of the pressure jump across a shock. To render the results more explicit, the basic theory will be developed principally for a nearly planar wing, of which the thickness and the surface lift distributions are prescribed. (The mean wing surface can be determined from the lift distribution. See § 3.2, (3.18*b*).)

To avoid problems of non-uniformity arising from edge singularities, both lift and thickness distributions will be assumed to vanish sufficiently smoothly

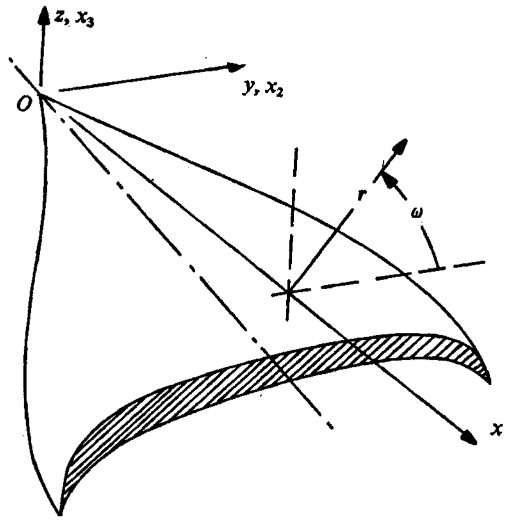


FIGURE 1. Illustration of the Cartesian and cylindrical co-ordinates. The shaded area represents a typical cross-section of the wing. The smooth lift distribution assumed in the paper requires a dropped leading edge, as shown.

at the trailing edges *and* at the leading edges. This restriction has also the benefit of avoiding leading-edge separation and local shock waves, making the solution much more easily realizable.†

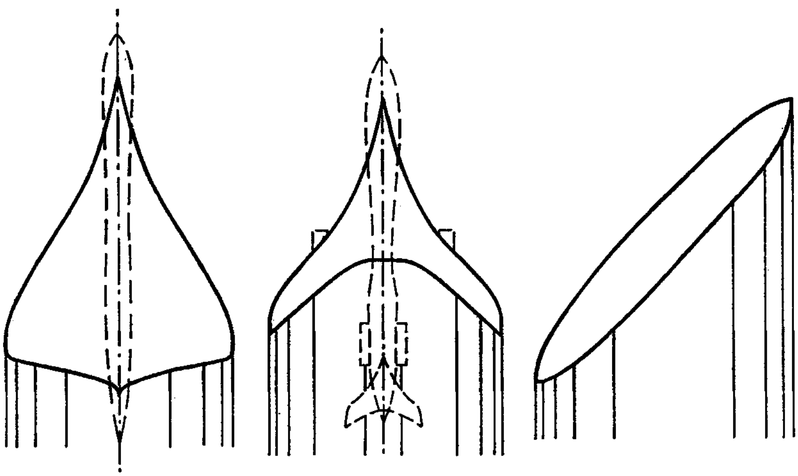
Viscous boundary layers and wakes are important aspects of transonic flow not treated in this paper. Ryzhov (1965) showed that the inviscid description of the three-dimensional far field breaks down at $M_\infty = 1$. But the effect of this far-field non-uniformity on the flow region analysed turns out to be exceedingly weak at Reynolds numbers in the practical range (Szaniawski 1968).

In the subsequent analysis, x, y and z denote Cartesian co-ordinates, with the axis pointing downstream, and the z axis in the lift direction. Alternatively, y and z may be replaced by the cylindrical polar co-ordinates r and ω, r denoting the distance from the x axis and ω the azimuthal angle (figure 1). The lift of the entire wing surface upstream of the station x will be denoted by $F^*(x)$. Often used also are the symbols b, l and $S_c^*(x)$, which represent the half-span, a length in the wind direction, and the local cross-sectional area cut by a plane transverse to the x axis, respectively. The lift and thickness distributions, as well as the leading-edge contour of the planform, are assumed to be so smooth that the length scales characterizing the axial and transverse field gradients in the vicinity of the wing are no less than l and b, respectively. The subsequent work will also require that the derivatives of $F^*(x)$ and $S^*(x)$ be continuous. The types of admissible planform are illustrated in figure 2.

The basic partial differential equation governing the perturbation (velocity) potential ϕ, written in Cartesian variables for the present purpose, is

$$(1-M_\infty^2)\,\phi_{xx}+\phi_{yy}+\phi_{zz} = U^{-1}M_\infty^2\frac{\partial}{\partial x}\left[\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\phi_x^2+\phi_y^2+\phi_z^2\right] + \text{cubic terms} \quad (2.1)$$

† A slight leading-edge bluntness is admissable in the theory.



(a) (b) (c)

FIGURE 2. Three types of idealized planforms, to which the present theory is applicable: (a) Concorde, (b) swept, (c) oblique. The result holds as long as the body width is small compared with the wing span. The body accompanying the skew wing of sketch (c) is not shown. (See Jones 1971.)

(cf. Lighthill 1954, pp. 361, 458). U and M_∞ are the undisturbed flow speed and Mach number, respectively; subscripts x, y and z signify partial derivatives. The pressure p can be computed from the perturbation velocity potential via the Bernoulli relation

$$(p/p_\infty)^{-1/\gamma} = 1-(\gamma-1)\,a_\infty^{-2}\,[U\phi_x+\tfrac{1}{2}(\nabla\phi)^2]. \tag{2.2}$$

For simplicity, the specific heat ratio γ will be assumed constant throughout the entire development. Since it appears in (2.1) only through the variation of the speed of sound, the principal nonlinear result obtained is applicable also to a more general gas, γ being interpreted as $1+[\rho a^{-2}(\partial^2 p/\partial\rho^2)_s]_\infty$.

If one writes the surface of the wing as $z = Z(x,y)$, the impermeability requirement on either side of the surface is

$$\phi_z = \left[(U+\phi_x)\frac{\partial}{\partial x}+\phi_y\frac{\partial}{\partial y}\right] Z\,(x,y). \tag{2.3}$$

The same equation applies to the trailing vortex sheet (or any stream surface) and relates the geometry of the sheet to the velocity field in its vicinity. (The latter equation implies a continuous upwash across the sheet only in the linear case.) The pressure difference across a trailing vortex sheet is required to vanish. Except in the vicinity of the trailing vortex sheet and an afterbody, the perturbation potential is required to vanish at points far from the wing.

The analysis will involve developments in two ranges of the transverse radius $r \equiv (y^2+z^2)^{\frac{1}{2}}$. In the first, corresponding to the inner region $r = O(b)$, the basic solution is governed by the linear transonic equation

$$\phi_{yy}+\phi_{zz} = 0.$$

This will be referred to as the Jones (1946) equation. The successive approximations involve nonlinear as well as non-transonic corrections (cf. (2.1)). The second range of r corresponds to the outer (nonlinear) region; its precise order depends on the lift, and will be brought out in the course of the analysis (§ 2.3). Unlike most earlier theories of the transonic equivalence rule based on the slender-body approximation, the ratio b/l will not be assumed to be small. (A more precise requirement on the range of b/l is given in § 2.3.†)

2.2. *Importance of nonlinear corrections*

The degree of asymmetry in the flow far from the body is primarily controlled by the lift in the form of a line doublet, as is apparent from the multipole expansion cited in § 1. There is, as mentioned earlier, a lift control of the outer flow of equal importance in the form of a line source. The following discussion will make evident that this additional equivalent source arises from second-order nonlinear corrections to the Jones (1946) solution. The discussion will stipulate that the Jones solution and its higher approximations are applicable in the vicinity of the wing.

Compressibility corrections. The terms written out on the right of (2.1),

$$U^{-1}M_\infty^2 \frac{\partial}{\partial x}\left[\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\phi_x^2+\phi_y^2+\phi_z^2\right],$$

represent the second-order compressibility corrections to the velocity divergence, and hence may be interpreted as a *distributed source* in the irrotational flow next to the body. Far enough from the wing (or the x axis), this distributed source, if integrable, should produce an effect equivalent to that of a line source, the strength of which is determined by the sum of the distributed source in the plane of constant x (of unit depth). This line source does not vanish with the wing thickness as long as lift persists, because the velocity squares contributing to the source lack the skew symmetry with respect to z. At sufficiently high lift, the source strength produced in this manner may even be greater than that corresponding to the geometrical cross-sectional area $S_c(x)$.

The two quadratic terms $\partial(\phi_y^2+\phi_z^2)/\partial x$ in (2.1) signify (twice) the rate of change of cross-flow kinetic energy, not accounted for in the classical transonic small-disturbance theory. Interestingly these terms correspond to the higher-order corrections treated by Van Dyke (1951) and Lighthill (1954) for supersonic flows past slender bodies of revolution, although their effect in that context is far less important. Significantly, these source-like terms give rise to a non-vanishing volume flux, producing an afterbody effect. Whereas the linear and nonlinear corrections to the inner solution will be determined unambiguously from the analysis in § 3, the contribution of $\partial(\phi_y^2+\phi_z^2)/\partial x$ in (2.1) to (the strength of) the equivalent line source may be expressed in terms of the rate of the total cross-flow kinetic energy without complicated analysis. Their contribution yields, accordingly, a total volume flux proportional to the vortex drag, corresponding to an equivalent afterbody with a cross-sectional area

$$S_1^*(\infty) = 2M_\infty^2(\text{vortex drag})/\rho_\infty U^2. \tag{2.4}$$

† For comparison of experiment with solutions based on Jones's equation, see e.g. Lock (1964).

Correction to inner boundary conditions. The foregoing delineation has not taken into account the nonlinear corrections at the inner boundary, which produce a comparable but opposite effect. To determine this additional equivalent source, one may evaluate the jump in upwash ϕ_z across the reference plane $z = 0$, which is compatible with the impermeability condition (2.3), and (2.1), on both sides of the stream surface. We shall adopt the symbols [ ] and $\langle\rangle$ to represent, respectively, the difference and the mean of values at the top and at the bottom of the surface. Application of (2.3) to both sides of the wing, taking the difference in ϕ_z across the wing, transferring the jump to the reference plane, and, finally, making use of (2.1) to eliminate $[\phi_{zz}]$, lead to

$$[\phi_z] = U\frac{\partial}{\partial x}[Z] + (1-M_\infty^2)\frac{\partial}{\partial x}\{[\phi_x]\langle Z\rangle + \langle\phi_x\rangle[Z]\}$$
$$+\frac{\partial}{\partial y}\{[\phi_y]\langle Z\rangle + \langle\phi_y\rangle[Z]\} + M_\infty^2\{[\phi_x]\langle Z_x\rangle + \langle\phi_x\rangle[Z_x]\} + \text{cubics.} \quad (2.5)$$

Integration on both sides of (2.5) with respect to y over the inner boundary gives an equivalent line source of a magnitude comparable to that arising from the compressibility correction examined earlier. Further integration over the entire x axis yields a total volume flux

$$\iint [\phi_z]\,dx\,dy = US_c^*(\infty) - M_\infty^2\,(\text{inviscid drag})/\rho_\infty U. \quad (2.6)$$

The inviscid drag is the sum of the drag due to the thickness and the drag due to lift.

The contribution of the thickness drag to the volume-flux balance cannot be too important, since $M_\infty^2\langle\phi_x\rangle[Z]_x$ in (2.5) is small compared with the linear term $U[Z]_x$. Therefore, the significant contribution of the inviscid drag in (2.6) comes primarily from the drag due to lift.

Resultant equivalent afterbody. Prior to and during the drag rise, the inviscid drag due to lift is contributed mainly by the vortex drag. In fact, so long as the inner flow region is not dominated by the thickness and can be kept shock-free, the drag associated with the shock loss in the outer flow is small compared with the vortex drag, as the subsequent study may confirm. Therefore, (2.4) and (2.6) may combine to yield a net volume flux resulting from the *second-order corrections*, giving an equivalent afterbody with a cross-sectional area (in physical variables)

$$S_e^*(\infty) = S_c^*(\infty) + M_\infty^2(\text{vortex drag})/\rho_\infty U^2. \quad (2.7)$$

Accordingly, a stream tube will enlarge its cross-section slightly after passing over a lifting wing, in the manner illustrated in figure 3. This and other properties of the source of the equivalent body will be substantiated later by the full theory in §§ 3 and 4.

The existence of an equivalent afterbody due to lift would not appear too surprising if one were to stipulate that the trailing vortex sheet lies flat in the plane $z = 0$. For such a model, any transverse motion in the 'Trefftz plane' would (according to Bernoulli's relation) make the pressure there, and hence the density, lower than that upstream. (Note that $M_\infty \neq 0$.) Thus a stream tube

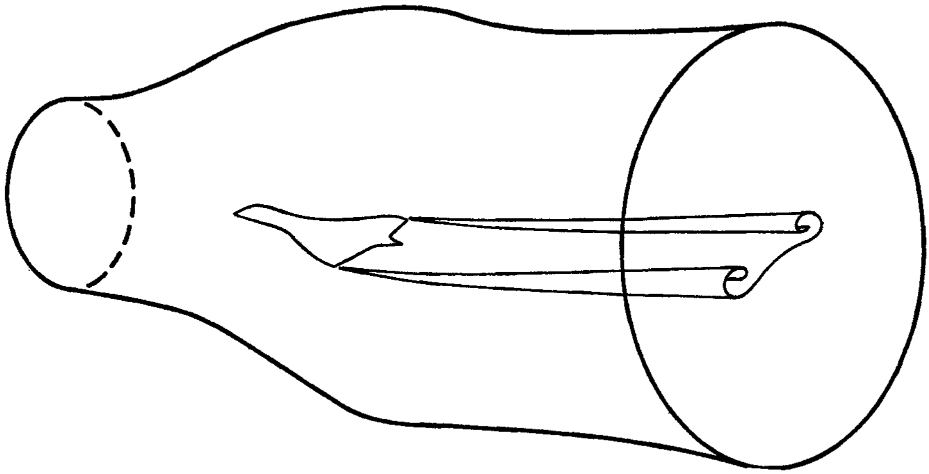


FIGURE 3. An afterbody effect produced by the lift. A stream tube enclosing an aircraft at transonic speed gains cross-sectional area by an amount proportional to the vortex drag. The widths of the cross-section and the trailing vortex sheet downstream are slightly exaggerated.

taken to be large enough to enclose the aircraft and its wake, but small enough to remain in the inner region (cf. figure 3) should have a larger cross-section far downstream (to conserve mass), as in the case with an afterbody. However, this interpretation is somewhat misleading (as pointed out by W. R. Sears and R. Seebass). Owing to the departure of the trailing vortex sheet from $z = 0$, there is a non-vanishing negative axial perturbation velocity far downstream (Sears 1974). Therefore, the pressure and density in question may not be lower than their corresponding upstream values. But the component of the *mass* flux intensity ρu computed from the Bernoulli equation is

$$\frac{\rho u}{\rho_\infty U} = 1 + (1 - M_\infty^2)\frac{\phi_x}{U} - \frac{M_\infty^2}{2}\frac{\phi_y^2 + \phi_z^2}{u^2} + O\left(\frac{\phi_x^2}{U^2}, \text{ cubics}\right). \tag{2.8}$$

Thus, in the *transonic* regime, the flux ρu far downstream is less than upstream, since M_∞ is close to unity and $\phi_x U^{-1}$ is as small as $\phi_z^2 U^{-2}$ there. Hence, the cross-sectional area of a stream tube (cut by the transverse plane) must be larger far downstream than upstream, in agreement with (2.7), irrespective of the sign of $(1 - M_\infty^2)$. In fact, from (2.8) one can recover $S_e^*(\infty)$ of (2.7).

We note that, in Hayes (1954), the cross-flow kinetic energy terms were omitted, on the grounds that wave systems tend to approach the planar, rendering their effects non-cumulative. This argument overlooks the contribution of these terms to the equivalent source that determines the initial data for the wave system. In addition, Hayes's (1954, equation (24)) would be inconsistent, were lift to dominate.

2.3. *Basic parameters and expansion procedures*

Basic to the present study are the parameters

$$\tau \equiv S_{\max}^*/bl, \quad \alpha \equiv F_{\max}^*/\rho_\infty U^2 b^2, \quad \lambda \equiv b/l \tag{2.9}$$

$S^*_{\max}$ and $F^*_{\max}$ are the maximum cross-sectional area and the maximum of the lateral force up to x, $F^*(x)$, respectively. Obviously, τ and α may be taken as the wing thickness and incidence parameters, respectively, characterizing typical flow angles in the inner region. The λ controls the leading-edge sweepback angle and may be referred to as the sweep parameter. It may be noted that λ is comparable with the classical planform aspect ratio ($\equiv 4b^2/(\text{wing area})$), where it is finite, but differs significantly from it if it is large!

Excluding complicated planforms,† the flow of a gas with a fixed γ over a wing may be characterized by four parameters: τ, α, M^2_∞ and λ. An alternative group of four is e.g. $\tau\lambda^3$, $\alpha\lambda^3$, $(M^2_\infty - 1)\lambda^2$ and $\tau\lambda$. The problem analysed thus involves multiple asymptotic limits. To render the subsequent formulation more definite, these parameters will be replaced by a third group ϵ, σ_*, K and Γ_*, where

$$\left.\begin{aligned} \sigma_* &\equiv (\gamma+1)^{\frac{1}{2}} M_\infty |\ln\epsilon|^{\frac{1}{2}} \alpha\lambda^{\frac{3}{2}}\tau^{-\frac{1}{2}}, \\ K \equiv (M^2_\infty - 1)\lambda^2\epsilon^{-2} \quad \text{and} \quad \Gamma_* &\equiv 8(\gamma+1)^{-1}\lambda^{-1}\lambda^{-2}|\ln\epsilon|^{-1}. \end{aligned}\right\} \qquad (2.10)‡$$

ϵ is the ratio of transverse length scales for the inner and the outer regions already mentioned, to be chosen according to either of the values

$$\epsilon = [(\gamma+1) M^2_\infty \tau\lambda^3]^{\frac{1}{3}}, \quad \epsilon|\ln\epsilon|^{-\frac{1}{2}} = (\gamma+1) M^2_\infty \alpha\lambda^3, \qquad (2.11\,a, b)$$

depending on the range of σ_* (cf. §3.1). The basis for the equivalence rule is the existence of a distinct inner region small compared with the outer region. This requires a small ϵ.

The basic formation will be developed for fixed, non-vanishing σ_*, Γ_*, and K in the *single* limit

$$\epsilon \to 0. \qquad (2.12)$$

Cases involving less restricted σ_*, Γ_* and K, including unbounded σ_*, will be examined thereafter. The cases of unbounded Γ_* and K correspond to slender wings and linear outer flows, respectively. They obey different scaling laws, and are excluded from the following analysis. In passing, we note that a non-vanishing or unbounded ϵ implies a high aspect-ratio planform with negligible sweep. It corresponds to the three-dimensional problem formulated by Cole (1969). (Also see Miles 1959, table 1.)

Obviously, K is a generalized transonic parameter. The first form of ϵ, (2.11a) and the corresponding K were in fact introduced by Drougge (1959), in his experimental study of the equivalence rule applied to twin bodies, and also by Berndt (1952, 1955) and Spreiter & Stahara (1971), with λ replaced by the aspect ratio. The parameters controlling the lift effects on the outer nonlinear flow are σ_* and Γ_*. In particular, the nonlinear lift contributions to the equivalence source will be controlled by σ^2_* and $\sigma^2_* \Gamma_*$. $\sigma^2_* \Gamma_*$ characterizes the part of the nonlinear correction not accounted for by the classical transonic small-disturbance equation, and is suggested readily by comparing the afterbody effect of (2.7) with $S^*_{\max}$. The factor $(\gamma+1) M^2_\infty$ in the definitions (2.10) and (2.11) is retained merely

† Yawed wings of extremely high aspect ratio will not be treated in this paper.

‡ For all practical purposes, a finite Γ_* may be taken to mean a finite and non-vanishing λ.

to conform with the traditional practice (Spreiter & Alksne 1958; Ashley & Landahl 1965; Cole 1969).† However, the inclusion of the factors $(\gamma+1)$ in (2.10) and (2.11) represents a definite gain; it eliminates the specific-heat ratio from the reduced outer equations. One may note in passing that σ_* is essentially the ratio of the two alternative values of ϵ given by (2.11); at $\sigma_* = 1$, the two ϵ's become identical.

2.4. *Logarithm in the inner expansion*

Expansions in powers of ϵ, with the logarithm of ϵ appearing in the coefficients, are familiar from classical work on flow past slender bodies (Broderick 1949; Lighthill 1954). A similar logarithm, often associated with 'switchback' terms, also arises in the low Reynolds number expansion. (See e.g. Van Dyke 1964, Lagerstrom & Casten 1972). When terms like $\epsilon^m |\ln \epsilon|^p$ are allowed in the inner expansion, question as to determinacy of the exponent p arises, since the inner system would admit additive (homogeneous) solutions satisfying the Jones equation for an arbitrary p. The correct choices for the powers of $\ln \epsilon$ are, of course, to be established by the final demonstration of the matched (asymptotic) expansions to the inner and outer problems. Nevertheless, a rule for inferring the logarithmic factor, without *a priori* knowledge of the full outer solution, is highly desirable, inasmuch as it simplifies the task of constructing the inner approximation and the analysis of its non-uniformity.

The rule in question stipulates that, with a uniform free stream, the outer solution is completely determined by the strength of the source, doublet and other singularities at the x axis. Thus, the part of the outer solution that is regular at the axis signifies a feedback from the far field that is controlled by the singularities mentioned. Hence, the order of terms $\epsilon^m|\ln \epsilon|^p$ in the regular part, if they appear, *cannot* be lower than those in the singular part of the outer solution. This rule, in turn, imposes a requirement on the harmonic functions $b_n(x)\, r^n \exp(in\omega)$ admissible to the inner expansion; in particular, on the logarithmic dependence of $b_n(x)$ on ϵ. To meet this requirement, we shall express the outer limit of the inner expansion in terms of the *outer* variable $\eta \equiv \epsilon r$, and eliminate those regular terms having coefficients proportional to $|\ln \epsilon|^p$ with p greater than those found in the singular part of the solution. For example, if the particular integral is

$$f(x) \ln^2 r = f(x)\,[\ln^2 \eta - 2 \ln \epsilon \ln \eta + \ln^2 \epsilon],$$

the proper particular solution consistent with the rule is

$$f(x)[-2 \ln \epsilon \ln \eta + \ln^2 \eta] = f(x)\,[\ln^2 r - \ln^2 \epsilon].‡$$

3. The inner problem

In this section, we shall determine the solution to the Jones equation and some of its higher-order approximations. Of importance to the subsequent work is their behaviour far from the x axis, where the inner solution ceases to be valid.

† The factor $(\gamma+1)\, M_\infty^2$ has been simplified from $2(1+\frac{1}{2}(\gamma-1)\, M_\infty^2)\, M_\infty^2$.

‡ It also proves convenient first to eliminate all such complementary solutions from the particular integral. (See §§3.1 and 3.2.)

An examination of the non-uniformity of these and still-higher approximations will then disclose the important parameters and proper scalings for the construction of the outer solution. To reveal more clearly the genesis of the single-limit expansion and its parameters noted in §2.3, we shall first examine a heuristic development of the inner solution based on an expansion in powers of τ, α and $(1-M_\infty^2)$.

3.1. *A heuristic development*

In terms of the reference length l, and the parameters α and τ, the top and bottom wing surfaces will be assumed representable in a dimensionless form

$$z/l = \alpha Z_0 \pm \tau Z_1. \tag{3.1a}$$

Z_0 and Z_1 are (differentiable) functions of x/L and y/b, independent of α, τ, λ and M_∞. Similarly, the differential pressure across the wing is assumed to be of the form

$$[p] = \rho_\infty U^2 \alpha\lambda [P]. \tag{3.1b}$$

$[P]$ is a function of x/l and y/b, independent of α, τ, λ and M_∞. We introduce the dimensionless inner variables

$$\tilde{x} \equiv x/l, \quad \tilde{y} \equiv y/b, \quad \tilde{z} \equiv z/b, \tag{3.2a}$$

as well as the complex variable ζ and its conjugate

$$\zeta \equiv \tilde{y} + i\tilde{z}, \quad \bar{\zeta} \equiv \tilde{y} - i\tilde{z}. \tag{3.2b}$$

To simplify typesetting work, the tildes will be omitted. For present purposes, (2.1) may be written, for $\varphi \equiv \phi/Ub$, as

$$\varphi_{\zeta\bar{\zeta}} = k(\varphi_x^2)_x + k\Gamma(\varphi_\zeta \varphi_{\bar{\zeta}})_x + \tfrac{1}{4}(M_\infty^2 - 1)\lambda^2 \varphi_{xx} + \ldots, \tag{3.3}$$

with $$k \equiv \tfrac{1}{8}(\gamma+1) M_\infty^2 \lambda^3, \quad \Gamma \equiv 8(\gamma+1)^{-1}\lambda^{-2}. \tag{3.4}$$

The jumps in the upwash and in the potential across the *reference* plane ($z = 0$), consistent with the assumed forms of wing surface and lift distribution (cf. (3.1) and (2.5)), are

$$[\varphi_z] = \tau[Z_1]_x + \alpha^2 k\Gamma[\varphi_0](Z_0)_x + \tfrac{1}{8}(\gamma+1)\alpha^2 k\Gamma^2(\varphi_{0y} Z_0)_y + \ldots, \tag{3.5}$$

$$[\varphi] = \alpha[\varphi_0] + \ldots, \tag{3.6}$$

where $$[Z_1] = 2Z_1 = [Z]/\tau l, \quad [\varphi_0] \equiv -\int_{-\infty} [P]\, dx.$$

The successive terms not written out are proportional to α^3 and $\tau\alpha$. Similarly, we have

$$\alpha(Z_0)_x = \langle \partial\varphi/\partial z \rangle + \ldots. \tag{3.6a}$$

These equations apply also to the trailing vortex sheet. Inspection of (3.3)–(3.6) suggests a form of $\phi/(Ub)$ in ascending powers of α, τ and $(1-M_\infty^2)$:

$$\phi/(Ub) \equiv \varphi = \alpha\varphi_0 + \tau\varphi_1 + \alpha^2 k\varphi_2 + \alpha^2 k\Gamma\psi_2 + \alpha^2 k\Gamma^2\psi_2' + \ldots \tag{3.7}$$

The coefficients φ_0, φ_1, etc., are expected to be *independent* of α, τ, $(1-M_\infty^2)\lambda^2$, as well as of k and Γ, except for a possible weak dependence through the logarithm of a scale factor (resulting from matching). Terms not written out in (3.7), as well as those omitted from (3.3)–(3.6a), belong to the *third*, and higher powers of the product of α, τ and $|M_\infty^2-1|^{\frac{1}{2}}$. Equation (3.3) yields others governing φ_0, φ_1, φ_2, etc.:

$$(\varphi_0)_{\zeta\bar{\zeta}} = 0, \tag{3.8a}$$

$$(\varphi_1)_{\zeta\bar{\zeta}} = (\varphi_2)_{\zeta\bar{\zeta}} - (\varphi_{0x}^2)_x = (\psi_2)_{\zeta\bar{\zeta}} - (\varphi_{0\zeta}\varphi_{0\bar{\zeta}})_x = (\psi_2')_{\zeta\bar{\zeta}} = 0. \tag{3.8b}$$

The inner boundary conditions for the coefficients $\varphi_0, \varphi_1, \ldots, \psi_2'$ follow from (3.5) and (3.6).

Although they will not affect the principal results, the far-field behaviour of terms belonging to the third- and the fourth-power groups mentioned are essential in securing a firm basis for the asymptotic theory to follow. For later reference, we list terms of the third-power group *in full*:

$$\alpha^3k^2\varphi_3 + \tau\alpha k\varphi_3' + \alpha(1-M_\infty^2)\lambda^2\varphi_3'' + \alpha^3k^2\Gamma\psi_3 + \alpha^3k^2\Gamma^3\psi_3' + \alpha^3k^2\Gamma^3\psi_3'' + \alpha^3k^2\Gamma^4\psi_3''' + \alpha\tau k\Gamma\psi_3^{(\mathrm{iv})} + \alpha\tau\Gamma^2\psi_3^{(\mathrm{v})}. \tag{3.9}$$

ψ_3''' and $\psi_3^{(\mathrm{v})}$ result from the remainder of (3.6). Parts of ψ_3, ψ_3', and ψ_3'' are contributed by cubic terms omitted from (3.3) or (2.1).† Typical among the fourth-power group are

$$\alpha^4k^3\varphi_4 + \tau\alpha^2k^2\varphi_4' + \tau^2k\varphi_4'' + \tfrac{1}{4}\tau(M_\infty^2-1)\varphi_4'' + \alpha^2k\tau(M_\infty^2-1)4^{-1}\varphi_4^{(\mathrm{iv})} + \alpha^4k^3\Gamma\psi_4 + \alpha^4k^3\Gamma^2\psi_4' + \ldots. \tag{3.10}$$

Typical of the partial differential equations governing terms of third- and fourth-power groups are

$$\left.\begin{aligned}
&(\varphi_3)_{\zeta\bar{\zeta}} - 2\frac{\partial}{\partial x}(\varphi_{0x}\varphi_{2x}) = 0,\\
&(\psi_3)_{\zeta\bar{\zeta}} - \frac{\partial}{\partial x}\left[2\varphi_{0x}\psi_{2x} + \varphi_{0\zeta}\varphi_{2\bar{\zeta}} + \varphi_{2\zeta}\varphi_{0\bar{\zeta}} + \frac{3-2\gamma}{3(\gamma+1)}\varphi_{0x}^3\right] = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\varphi_4)_{\zeta\bar{\zeta}} - \frac{\partial}{\partial x}(\varphi_{2x}^2 + 2\varphi_{0x}\varphi_{3x}) = 0,\\
&(\varphi_4'')_{\zeta\bar{\zeta}} - \frac{\partial}{\partial x}(\varphi_{1x}^2) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3.11}$$

A fuller description of the system is given in Cheng & Hafez (1973*a*, appendix 1.1).

With the procedure indicated in § 2.4, concerning $|\ln\epsilon|$, asymptotic behaviour of φ_0, φ_1, etc. at large $r = |\zeta|$ may be inferred quite readily from the partial differential equations and the inner boundary conditions. In particular, their r dependence may be written as

$$\left.\begin{aligned}
&\varphi_0 = r^{-1}, \quad \varphi_1 = \ln r + \ln\epsilon = \ln(\epsilon r),\\
&\varphi_2 = \ln^2 r - \ln^2\epsilon = 2\ln\epsilon\ln(\epsilon r) + \ln^2(\epsilon r),\\
&\psi_2 = 2^{-1}(\varphi_0^2)_x + \text{harmonic} = \ln r + \ln\epsilon = \ln(\epsilon r),\\
&\varphi_3 = r\ln\epsilon\ln^2(\epsilon r), \quad \varphi_3'' = r\ln(\epsilon r),\\
&\varphi_4 = r^2\ln^2\epsilon\ln(\epsilon r), \quad \varphi_4'' = r^2\ln^2(\epsilon r).
\end{aligned}\right\} \tag{3.12}$$

† The cubic terms not written out in (2.1) are

$$\tfrac{1}{2}(\gamma-1)a_\infty^{-2}(\phi_y^2+\phi_z^2)\phi_{xx} - (\gamma-1)a_\infty^{-1}\phi_x[(1+\tfrac{1}{2}(\gamma-1)M_\infty^2)\phi_x^2+\phi_y^2+\phi_z^2] + \tfrac{1}{2}\nabla\phi\,.\,\nabla(\nabla\phi)^2.$$

The order of magnitude signs have been omitted for convenience. In particular, the successive terms in (3.7),

$$\alpha\varphi_0; \quad \tau\varphi_1, \alpha^2 k\varphi_2, \alpha^2 k\Gamma\psi_2; \quad \alpha^3 k^2\varphi_3, \alpha(1-M_\infty^2)\lambda^2\varphi_3''; \quad \alpha(\alpha k)^3\varphi_4, \tau^2 k\varphi_4'', \tag{3.13a}$$

may now be arranged into the following form at *large r*, after division by τ, and replacing functions of x and ω by unity for simplicity:

$$\left.\begin{array}{c}\epsilon(\tau k)^{-\frac{1}{2}}\sigma_*\kappa^{-\frac{1}{2}}\eta^{-1}; \quad \ln\eta; \quad \sigma_*^2\ln\eta, \sigma_*^2\Gamma_*\ln\eta; \\ (\tau k)^{\frac{1}{2}}\epsilon^{-1}\sigma_*^3\kappa^{-\frac{1}{2}}\eta\ln\eta, \epsilon(\tau k)^{-\frac{1}{2}}\sigma_*\kappa^{-\frac{1}{2}}K\eta\ln\eta; \quad \tau k\epsilon^{-2}\sigma_*^4\eta^2\ln\eta, \tau k\epsilon^{-2}\eta^2\ln\eta.\end{array}\right\} \tag{3.13b}$$

For convenience, we have made the substitution

$$\kappa \equiv |\ln\epsilon|, \quad \eta \equiv \epsilon r. \tag{3.14}$$

This form of $\phi/(\tau Ub)$ clearly exhibits the appropriate grouping for the terms and the parameters σ_*, Γ_* and K of (2.10). It reveals at once the solution's parametric structure, and its relation to the range of validity of the inner solution. It is apparent from (3.13*b*) that, for finite σ_*, Γ_* and K, the expansion (3.7) will break down in the range of $r = O[(\tau k)^{-\frac{1}{2}}]$. For if one takes $\epsilon = (\tau k)^{\frac{1}{2}}$, the four successive groups of terms in (2.13*b*) become either $O(1)$ or $O(\kappa^{-\frac{1}{2}})$ as $\eta \equiv \epsilon r$ approaches unit order. On the other hand, for an unbounded σ_*, the inner expansion breaks down in another range corresponding to an $r = O(\epsilon^{-1})$, now with

$$\epsilon = \sigma_*(\tau k)^{\frac{1}{2}};$$

for the successive groups of terms in (3.13*b*) will then become either $O(\sigma_*^2)$ or $O(\sigma_*^2\kappa^{-\frac{1}{2}})$ as $\eta = \epsilon' r$ approaches order unity.†

The foregoing examination has not only revealed the important outer scale in two ranges of the lift parameter σ_*, but also suggests that the outer as well as the inner problems can be more systematically analysed by an expansion in small ϵ for the two ranges of σ_*, with fixed Γ and K. This is done below.

3.2. *The inner solution*

We shall formulate the inner problem on the basis of an expansion

$$\left.\begin{array}{c}\phi/(\alpha Ub) = \varphi_{\mathrm{I}} + \epsilon\varphi_{\mathrm{II}} + \epsilon^2\varphi_{\mathrm{III}} + \epsilon^3\varphi_{\mathrm{IV}} + \ldots, \\ \epsilon \equiv [(1+1)M_\infty^2\tau\lambda^3]^{\frac{1}{2}},\end{array}\right\} \tag{3.15}$$

in the single limit $\epsilon \to 0$, with fixed, non-vanishing parameters

$$\left.\begin{array}{c}\sigma_* \equiv (\gamma+1)^{\frac{1}{2}}M_\infty\alpha\lambda^{\frac{3}{2}}\tau^{-\frac{1}{2}}|\ln\epsilon|^{\frac{1}{2}}, \quad \Gamma_* \equiv 8(\gamma+1)^{-1}\lambda^{-2}|\ln\epsilon|^{-1}, \\ K \equiv (M_\infty^2-1)\lambda^2\epsilon^{-2}.\end{array}\right\} \tag{3.15a}$$

The cases involving less restricted σ_*, Γ_* and K, including unbounded σ_*, will be studied subsequently. As in the classical slender-body theory (cf. §2.4), logarithmic dependence of φ_{II}, φ_{III}, etc. on ϵ, in the form of powers of $\kappa^{-1} \equiv |\ln\epsilon|^{-1}$, is allowed, but under *no* circumstances will $|\ln\epsilon|^{-1}$ be treated as a negligible quantity. The expansion (3.15) satisfying (3.3) may be expressed in terms of

† It also follows from (3.13*b*) that the terms representing the feedback (cf. §2.4) to be allowed in φ_1, φ_2, etc. cannot have powers of $\ln\epsilon$ higher than those shown in each of (3.12).

φ_0, φ_1, etc. of (3.7), which satisfy individual sets of partial differential equations and inner boundary conditions (cf. (3.8), (3.5) and (3.6)). With

$$\sigma_1 \equiv \sigma_* |\ln \epsilon|^{-\frac{1}{2}}, \quad \Gamma \equiv \Gamma_* |\ln \epsilon|, \tag{3.16}$$

one finds
$$\varphi_{\mathrm{I}} = \varphi_0, \quad \varphi_{\mathrm{II}} = \sigma_1^{-1}\varphi_1 + \tfrac{1}{8}\sigma_1(\varphi_2 + \Gamma\psi_2 + \Gamma^2\psi_2'), \tag{3.17a, b}$$
$$\varphi_{\mathrm{III}} = \tfrac{1}{64}\sigma_1^2(\varphi_3 + \Gamma\psi_3 + \Gamma^2\psi_3' + \Gamma^3\psi_3'' + \Gamma^4\psi_3''') + \tfrac{1}{8}(\varphi_3' + \Gamma\psi_3^{(\mathrm{iv})} + \Gamma^2\psi_3^{(\mathrm{v})}) + \tfrac{1}{4}K\varphi_3'',$$
$$\varphi_{\mathrm{IV}} = \tfrac{1}{512}\sigma_1^3\varphi_4 + 8^{-2}\sigma_1\varphi_4' + \sigma_1^{-1}\varphi_4'' + \tfrac{1}{4}\sigma_1^{-1}K\varphi_4''' \tag{3.17c}$$
$$+ \tfrac{1}{32}\sigma_1 K\varphi_4^{(\mathrm{iv})} + \Gamma[\tfrac{1}{8}\sigma_1^3(\psi_4 + \Gamma\psi_4') + \ldots] + \ldots. \tag{3.17d}$$

φ_{III} and φ_{IV}, identified with the third- and fourth-power groups of (3.9)and (3.10), will be needed later only for an unambiguous estimate of their orders of magnitudes at large r. Equations (3.15), with (3.17), are none other than the heuristic expansion (3.7) recast in terms of σ_1, K, Γ and ϵ.†

The solutions φ_0 and φ_1, corresponding to the purely lifting and thickness problems in the slender-body theory, are (see e.g. Adams & Sears 1953)

$$\varphi_0 = \mathrm{Re}\,(2\pi i)^{-1}\int_{a_1}^{a_2} [\varphi_0(x, y_1)]\,(y_1 - \zeta)^{-1}\,dy_1, \tag{3.18}$$

$$\varphi_1 = \mathrm{Re}\,(2\pi)^{-1}\int_{a_1}^{a_2} [Z_1(x, y_1)] \ln[\epsilon(\gamma_1 - \zeta)]\,dy_1 + b_1(x). \tag{3.19}$$

a_1 and a_2 are the spanwise ordinates (divided by the half-span) of the two outermost edges of the wing and trailing vortex sheet. Of importance is their behaviour at large $r = |\zeta|$:

$$\varphi_0 \sim (2\pi)^{-1}F(x)\,r^{-1}\sin\omega + (2\pi)^{-1}m_{32}r^{-2}\sin 2\omega, \tag{3.18a}$$
$$\varphi_1 \sim (2\pi)^{-1}S_c'(x)\ln(\epsilon r) + b_1(x) - (2\pi)^{-1}(\bar{y}S_c)'\,r^{-1}\cos\omega, \tag{3.19a}$$

where
$$\left.\begin{aligned} F(x) &\equiv \int_{a_1}^{a_2}[\varphi_0]\,dy, \quad m_{32}(x) \equiv \int_{a_1}^{a_2}[\varphi_0]\,y\,dy, \quad S_c'(x) \equiv dS_c/dx, \\ S_c(x) &\equiv \int_{a_1}^{a_2}[Z_1]\,dy, \quad \bar{y}S_c(x) \equiv \int_{a_1}^{a_2}[Z_1]\,y\,dy. \end{aligned}\right\} \tag{3.20}$$

$F(x)$ and $S_c(x)$ signify the lift and the cross-sectional area made dimensionless by their respective *maximum* values (cf. definitions of α and τ, §2.3). Introduction of the $\ln\epsilon$ in (3.19) and (3.19a) is in accord with the procedure stated in §2.4. We note in passing that Z_0 and $[\varphi_0]$ are related through (3.6a), i.e.

$$\frac{\partial}{\partial x}Z_0(x, y) = \left(\frac{\partial}{\partial z}\varphi_0\right)_{z=0} = (2\pi)^{-1}\,\mathrm{p.v.}\int_{a_1}^{a_2}(y_1 - y)^{-1}\frac{\partial}{\partial y_1}[\varphi_0]\,dy_1. \tag{3.18b}$$

The solution to the second of (3.8b), which is real and single-valued, with continuous φ_2 and $\partial\varphi_2/\partial z$ across the wing, consistent with the boundary conditions (3.5) and (3.6), is obtained explicitly:

$$\begin{aligned} \varphi_2 = &-(8\pi^2)^{-1}y\frac{\partial}{\partial x}\int_{a_1}^{a_2}[\varphi_0(x, y_1)]_x\,dy_1\int_{a_1}^{a_2}\frac{[\varphi_0(x_1 y_2)]_x}{y_2 - y}\ln\frac{(y_2 - \zeta)(y_2 - \bar{\zeta})}{(y_1 - \zeta)(y_1 - \bar{\zeta})}\,dy_2 \\ &+ (16\pi^2)^{-1}\frac{\partial}{\partial x}\left\{\int_{a_1}^{a_2}[\varphi_0(x, y_1)]_x \ln[(y_1 - \zeta)(y_1 - \bar{\zeta})]\,dy_1\right\}^2 + (2\pi)^{-1}\frac{\partial}{\partial x}\int_{a_1}^{a_2}\chi(x, y_1) \\ &\times \ln[\epsilon^2(y_1 - \zeta)(y_1 - \bar{\zeta})]\,dy_1 - (4\pi^2)^{-1}(F_x^2)_x(\ln^2\epsilon + \tfrac{1}{2}) + b_2(x)|\ln\epsilon|. \end{aligned} \tag{3.21}$$

† The pressure coefficient $C_p \equiv 2(p - p_\infty)/\rho_\infty U^2$ may be computed from

$$C_p/(\alpha\lambda) = -2\partial\phi_{\mathrm{I}}/\partial x - \epsilon\{2\partial\phi_{\mathrm{II}}/\partial x + J[(\partial\phi_{\mathrm{I}}/\partial y)^2 + (\partial\phi_{\mathrm{I}}/\partial z)^2]\} + O(\epsilon^2),$$

with $J \equiv 8^{-\frac{1}{2}}(\gamma + 1)^{\frac{1}{2}}M_\infty^{-1}\sigma_1\Gamma^{\frac{1}{2}}$.

Here, $\chi(x,y) \equiv -(\pi)^{-1}[\varphi_0(x,y)]_x \,\mathrm{p.v.}\int_{a_1}^{a_2} [\varphi_0(x,y_1)]_x \left[\frac{y}{y_1-y} + \ln|y_1-y|\right] dy_1$

results from a source distribution introduced over the wing plane to make $\partial\varphi_2/\partial z$ continuous. Intermediate steps of the derivation, given in Cheng & Hafez (1973*a*, appendix 1.2), are omitted for brevity. Terms proportional to $|\ln\epsilon|^2$ and $|\ln\epsilon|$ among the integration constants have been introduced above so that, in the variable $\eta \equiv \epsilon r$, the regular and singular terms possess the same logarithmic dependence on ϵ (cf. § 2.4). Equation (3.21) yields the crucial behaviour for φ_2 at large r

$$\varphi_2 \sim (8\pi^2)^{-1}(F_x^2)_x[2\ln^2(\epsilon r) + \cos 2\omega] + (2\pi^2)^{-1}\frac{d}{dx}\left[F_x^2|\ln\epsilon| + \pi\int_{a_1}^{a_2}\chi(x,y)\,dy\right]\ln(\epsilon r)$$
$$+ b_2(x)|\ln\epsilon| + (4\pi^2)^{-1}\frac{d}{dx}\left[F_x(m_{32})_x(2\ln\epsilon+1) - 2\pi\int_{a_1}^{a_2} y^2\chi\,dy\right] r^{-1}\cos\omega$$
$$+ (4\pi^2)^{-1}[F_x(m_{32})_x]_x\, r^{-1}[2\ln(\epsilon r)\cos\omega - \cos 3\omega] + \dots. \qquad (3.21a)$$

The second and the fourth terms signify an equivalent line source and a weaker equivalent line doublet in the far field, which result from nonlinear corrections to the inner solution.

Similar, and perhaps even more significant, is the source-like contribution of ψ_2 associated with the cross-flow kinetic energy, unaccounted for by the transonic small-disturbance theory (cf. § 2.2). The third of (3.8*b*) governing ψ_2 has an obvious particular integral

$$\psi_2 = \tfrac{1}{2}(\varphi_0^2)_x,$$

which is equivalent to the second of the three particular integrals in Lighthill (1954, p. 463) cited earlier. But this solution gives an upwash discontinuity in the wing plane, which does not agree with that imposed by (3.5), i.e.

$$[\partial\psi_2/\partial z] = [\varphi_0]_x\,\partial Z_0/\partial x \quad \text{or} \quad [\varphi_0]_x\langle\partial\varphi_0/\partial z\rangle.$$

The required solution is

$$\psi_2 = \tfrac{1}{2}(\varphi_0^2)_x - \mathrm{Re}\,(2\pi)^{-1}\int_{a_1}^{a_2}\varphi_0\frac{\partial^2}{\partial x^2}Z_0\ln[\epsilon(y_1-\zeta)]\,dy_1 + \tilde{b}_2(x). \qquad (3.22)$$

This appears to be different from the corresponding result of Cheng & Hafez (1973*a*, *b*), because there the contribution of $[\partial\psi/\partial z]$ enters through φ_1 instead of ψ_2. The solution yields a line source and a (weaker) line doublet

$$\psi_2 \sim (2\pi)^{-1}E'(x)\ln(\epsilon r) + \tilde{b}_2(x) - (2\pi)^{-1}\frac{d}{dx}\int_{a_1}^{a_2}[\varphi_0]\frac{\partial^2}{\partial x^2}Z_0 y\,dy\,r^{-1}\cos\omega, \quad (3.22a)$$

with $$E'(x) \equiv -\int_{a_1}^{a_2}[\varphi_0]\frac{\partial^2}{\partial x^2}Z_0\,dy = -\frac{1}{2\pi}\iint[\varphi_0]_{xy}[\varphi_0]_{y_1}\ln|y-y_1|\,dy\,dy_1, \quad (3.23)$$

to be identified with the rate of change of the cross-flow kinetic energy in § 5.2.

Finally, ψ_2' is a harmonic function satisfying the jump condition specified by the last term in (3.5)

$$\psi_2' = \mathrm{Re}\,(2\pi)^{-1}\int_{a_1}^{a_2}\tfrac{1}{8}(\gamma+1)\frac{\partial}{\partial y_1}\left\{\frac{\partial}{\partial y_1}[\phi_0]Z_0\right\}\ln[\epsilon(y_1-\zeta)]\,dy_1. \qquad (3.24)$$

This yields a line doublet at large r

$$\psi_2' \sim (16\pi)^{-1}(\gamma+1)\int_{a_1}^{a_2}[\varphi_0]_y Z_0 dy\, r^{-1}\cos\omega. \tag{3.24a}$$

For the purpose of analysing the non-uniformity of the expansion (3.15), it suffices to infer the order of magnitude of the functions φ_3, φ_3', etc. from the particular integrals of their governing partial differential equations, making use of the foregoing asymptotic forms of $\varphi_0, \varphi_1, \ldots, \psi_2'$, and following the procedure in selecting the $\ln\epsilon$ terms (cf. §2.4). The results obtained for the third-order coefficients, omitting the azimuthal dependence as well as the order of magnitude signs, are (cf. (3.9))

$$\left.\begin{aligned} &\varphi_3 = r\ln^2(\epsilon r), \quad \varphi_3' = r\ln^2(\epsilon r), \quad \varphi_3'' = r\ln(\epsilon r),\\ &\psi_3 = r\ln^2(\epsilon r), \quad \psi_3' = r^{-1}, \quad \psi_3'' = r^{-1}, \quad \psi_3''' = r^{-1},\\ &\psi_3^{(\mathrm{iv})} = r^{-1}, \quad \psi_3^{(\mathrm{v})} = r^{-1}. \end{aligned}\right\} \tag{3.25}$$

Typical among the fourth-order coefficients are (cf. (3.10))

$$\left.\begin{aligned} &\varphi_4 = r^2\ln^2\epsilon\ln(\epsilon r), \quad \varphi_4' = r^2\ln\epsilon\ln^2(\epsilon r), \quad \varphi_4'' = r^2\ln^2(\epsilon r),\\ &\varphi_4''' = r^2\ln(\epsilon r), \quad \varphi_4^{(\mathrm{iv})} = r^2\ln\epsilon\ln(\epsilon r), \quad \psi_4 = r^2\ln\epsilon\ln^2(\epsilon r),\\ &\psi_4' = r^2\ln^2(\epsilon r), \ldots. \end{aligned}\right\} \tag{3.26}$$

Note that the skew-symmetry of the third-order solutions with respect to z does not permit terms of order unity in the coefficients of (3.25); but a feedback in the form of upwash $b_3(x)\, r\exp(i\omega)$ must be allowed, which can be absorbed under any of φ_3, φ_3', φ_3'' and ψ_3.

Instead of (3.15), an alternative expansion is possible:

$$\phi/(\alpha Ub) = \varphi_{\mathrm{I}}' + \epsilon'\varphi_{\mathrm{II}}' + \epsilon'^2\varphi_{\mathrm{III}}' + \epsilon'^3\varphi_{\mathrm{IV}}' + \ldots, \quad \epsilon'|\ln\epsilon|^{-\frac{1}{2}} = (\gamma+1)M_\infty^2\lambda^3\alpha. \tag{3.27}$$

The prime on ϵ indicates the second definition for ϵ given in (2.11*b*). The functions φ_{I}', φ_{II}', etc., may again be evaluated in terms of φ_0, φ_1, etc. via

$$\varphi_{\mathrm{I}}' = \varphi_0, \quad \varphi_{\mathrm{II}}' = \sigma_*^{-1}\varphi_{\mathrm{II}}, \quad \varphi_{\mathrm{III}}' = \sigma_*^{-2}\varphi_{\mathrm{III}}, \quad \varphi_{\mathrm{IV}}' = \sigma_*^{-3}\varphi_{\mathrm{IV}}, \tag{3.27a}$$

with ϵ' replacing ϵ in the definitions of σ_*, Γ_* and K (cf. (3.15*a*) or (2.10)). For finite and non-vanishing σ_*, Γ_* and K, the two expansions (3.27) and (3.15) are equivalent and, in fact, identical at $\sigma_* = 1$. But (3.27) yields valid results for an *unbounded* σ_*, for which (3.15) breaks down (at least formally).

3.3. *The non-uniformity*

The expansion of ϕ in ϵ (3.15) may now be written at large r, using results obtained for φ_0, φ_1, φ_2, etc., and writing κ for $|\ln\epsilon|$ and η for ϵr:

$$\begin{aligned} \phi/(\tau Ub) = \sigma_*\varphi/(\alpha\epsilon|\ln\epsilon|^{\frac{1}{2}}) \sim{}& (2\pi)^{-1}\{\kappa^{-\frac{1}{2}}\sigma_* F(x)\,\eta^{-1}\sin\omega + O(\epsilon\eta^{-2})\}_{\mathrm{I}}\\ &+ (2\pi)^{-1}\{S_e'(x)\ln\eta + b_1(x) + \tfrac{1}{8}\sigma_*^2[\Gamma_*\tilde{b}_2(x) + b_2(x)]\\ &+ (32\pi)^{-1}\sigma_*^2\kappa^{-1}(F_x^2)_x(2\ln^2\eta + \cos 2\omega) + O[\epsilon(1+\sigma_*^2)\eta^{-1}]\}_{\mathrm{II}}\\ &+ O\{\kappa^{-\frac{1}{2}}\sigma_*^3(1+\Gamma_*)\,\eta\ln^2\eta + \kappa^{-\frac{1}{2}}\sigma_*\,\eta\,(\ln^2\eta + K\ln\eta)\}_{\mathrm{III}}\\ &+ O\{[\sigma_*^4 + 1 + K(1+\sigma_*^2) + \sigma_*^4\Gamma_*(1+\Gamma_*)]\,(\eta^2\ln\eta + 1)\}_{\mathrm{IV}}. \end{aligned} \tag{3.28}$$

S'_e is the strength of a line source corresponding to the wing

$$S'_e(x) \equiv \frac{d}{dx}\left\{S_c(x) + (8\pi)^{-1}\sigma_*^2\left[F_x^2 + \kappa^{-1}\pi\int_{a_1}^{a_2}\chi\, dy + \pi\Gamma_* E(x)\right]\right\}. \tag{3.28a}$$

The Roman numeral subscripts I, II, III and IV serve to identify φ_{I}, φ_{II}, φ_{III} and φ_{IV}, respectively. The terms $\ln^2\eta$ and $\ln\eta$ inside the brackets, with subscripts III and IV, stand for terms $O(\ln^2\eta)$ and $O(\ln\eta)$, as well as $O(1)$. The latter allows for the feedback terms mentioned in §2.4. As $\eta \equiv \epsilon r$ approaches order unity with fixed σ_*, Γ_* and K, the *even* Roman numeral groups corresponding to $\epsilon\varphi_{\mathrm{II}}$ and $\epsilon^3\varphi_{\mathrm{IV}}$ tend to unit order, whereas the *odd* numeral groups corresponding to φ_{I} and $\epsilon^2\varphi_{\mathrm{III}}$ tend to $O(\kappa^{-\frac{1}{2}})$, i.e. the order of $|\ln\epsilon|^{-\frac{1}{2}}$. An examination of terms belonging to still higher orders confirms this trend. Hence, the non-uniform region for the inner expansion (3.15) is unambiguously identified with $\epsilon r = O(1) \neq 0$. The region of validity for the far-field description of the inner solution (3.28) is therefore established at

$$\epsilon \ll \eta \ll 1 \quad \text{with} \quad \eta \equiv \epsilon r = [(\gamma+1)M_\infty^2\tau\lambda^3]^{\frac{1}{2}} r, \tag{3.29}$$

as long as σ_*, Γ_* and K are finite.

The explicit dependence of K, Γ_* and σ_* in (3.28) shows clearly that the latter's non-uniformity remains unchanged, even if K, Γ_* and σ_* vanish *with* ϵ. Thus, the analysis includes the non-lifting problem ($\sigma_* = 0$). But this expansion in ϵ is inappropriate for an unbounded σ_*, as is apparent from (3.28).

For unbounded σ_*, the ϵ' expansion of (3.27) is applicable, and yields the asymptotic result at large r

$$\begin{aligned}\phi/(\gamma+1)M_\infty^2\alpha^2\lambda^3|\ln\epsilon'|Ub \sim{}& (2\pi)^{-1}\{\kappa^{-\frac{1}{2}}F(x)(\eta')^{-1}\sin\omega + O(\epsilon'(\eta')^2)\}_{\mathrm{I}}\\ &\times(2\pi)^{-1}\{\hat{S}'_e\ln\eta' + 8^{-1}(\Gamma_*\tilde{b}_2 + b_2) + b_1\sigma_*^{-2} + (32\pi)^{-1}\kappa^{-1}(F_x^2)_x(2\ln^2\eta + \cos 2\omega)\\ &+ O[\epsilon(1+\sigma_*^{-2})(\eta')^{-1}]\}_{\mathrm{II}} + O\{\kappa^{-\frac{1}{2}}(1+\Gamma^*)\eta'\ln^2\eta' + \kappa^{-\frac{1}{2}}\eta'(\sigma_*^{-2}\ln^2\eta' + K\ln\eta')\}_{\mathrm{III}}\\ &+ O\{(\eta'^2\ln^2\eta' + 1)[1 + \sigma_*^{-2}K(1+\sigma_*^{-4}) + \Gamma_*(1+\Gamma_*)\}_{\mathrm{IV}},\end{aligned} \tag{3.30}$$

where

$$\hat{S}'_e = S'_e\sigma_*^{-2}, \quad \eta' \equiv \epsilon' r. \tag{3.30a}$$

ϵ' has been substituted for ϵ in the definitions of κ, σ_*, Γ_* and K. The result corresponds, term by term, to the ϵ expansion at large r (3.28). In fact, the two become identical for $\sigma_* = 1$, as noted earlier. As $\eta' \equiv \epsilon' r$ approaches order unity, the even and odd Roman numeral groups of (3.31) are seen to tend to $O(1)$ and $O(\kappa^{-\frac{1}{2}})$ respectively, for *bounded* K, Γ_* and σ_*^{-1}, including vanishing K and Γ_*, and $\sigma_* \to \infty$. The non-uniform region in this case is therefore identified with $\epsilon' r = O(1) \neq 0$; and the region of validity for the far-field description of the inner ϵ' expansions (3.30) is

$$\epsilon' \ll \eta \ll 1 \quad \text{with} \quad \eta' \equiv \epsilon' r \quad \text{and} \quad \epsilon'|\ln\epsilon'|^{-\frac{1}{2}} = (\gamma+1)M_\infty^2\alpha\lambda^3. \tag{3.31†}$$

† The estimates for the third-order group given in (3.28) and (3.30) differ slightly from our previous result in that terms independent of $\kappa^{-\frac{1}{2}}$ found in Cheng & Hafez (1973*a*, *b*) are absent. As a result $\epsilon^2\phi_{\mathrm{III}}$ in our earlier work becomes to $O(1)$ instead of $O(\kappa^{-\frac{1}{2}})$ when η or η' approaches $O(1)$. The estimates given for ψ'_3 and ψ''_3 (which is $\psi^{(\mathrm{IV})}$ in this paper) were also in error, but of little consequence.

4. The nonlinear outer region

4.1. *Three nonlinear domains*

The results (3.28) and (3.30) describe the inner solution in its outer limit for bounded as well as unbounded σ_*. They indicate that the field far from the axis is determined principally by a line doublet and a line source of strengths, subject to a correction of $O(\epsilon)$,

$$\left.\begin{aligned} D_0(x) &= \sigma_*|\ln\epsilon|^{-\frac{1}{2}}F(x) = \sigma_1 F(x), \\ S_e'(x) &= \frac{d}{dx}\left\{S_c(x) + (8\pi)^{-1}\sigma_*^2\left[F_x^2 + \kappa^{-1}\pi\int_{a_1}^{a_2}\chi\,dy + \pi\Gamma_* E(x)\right]\right\}, \end{aligned}\right\} \tag{4.1}$$

respectively. The relative importance of the lift and thickness in determining $D_0(x)$ and $S_e'(x)$, and hence the outer flow, is seen to be controlled by

$$\sigma_*^2 \equiv (\gamma+1)\,M_\infty^2|\ln\epsilon|\,\alpha^2\lambda^3\tau^{-1},$$

for given Γ_* and K. Three domains of lift control on the outer flow are apparent: (I) thickness-dominated, $\sigma_* \ll 1$; (II) intermediate, $\sigma_* = O(1)$; (III) lift-dominated, $\sigma_* \gg 1$. The first domain corresponds to that of the classical transonic area rule, although departure from the axisymmetry represented by the line doublet distribution $D_0(x)$ is not always negligibly small. The third domain allows for wings with zero thickness. The lift control on the equivalent source S_e' is generally characterized by σ_*^2. The part in S_e' representing a departure from the classical transonic small-disturbance theory is characterized by $\sigma_*^2\,\Gamma_* = \sigma_1^2\,\Gamma$.

The case of an unbounded Γ_* corresponding to a slender wing ($\lambda \ll 1$) will not be treated fully in this paper. But it may be pointed out that the proper lift-control parameter in this case is $\sigma_1^2\Gamma$, rather than σ_*, and that the present theory may still be applied to slender wings, so long as σ_1 and $\sigma_1^2\Gamma$ are finite.

4.2. *Two sets of variables*

It is apparent from the analysis in §3.3 that a proper formulation of the outer problem for an *unrestricted* σ_*, with finite Γ_* and K, will require two sets of variables:

$$x, \eta \equiv \epsilon r, \omega; \quad \Phi \equiv \phi/(\tau U b); \quad \epsilon \equiv [(\gamma+1)\,M_\infty^2\,\tau\lambda^3]^{\frac{1}{2}}; \tag{4.2a}$$

$$x, \eta' \equiv \epsilon' r, \omega; \quad \Phi' \equiv \phi/[\alpha^2\lambda^3(\gamma+1)\,M_\infty^2|\ln\epsilon'|bU]; \quad \epsilon'|\ln\epsilon'|^{-\frac{1}{2}} = (\gamma+1)\,M_\infty^2\alpha\lambda^3. \tag{4.2b}$$

As before, x and r are dimensionless variables based on the length scales l and b, respectively. Obviously, the first set (4.2*a*) applies to domains I and II, corresponding to finite σ_*, and the second set (4.2*b*) to domains III and also II, corresponding to finite σ_*^{-1}. It is useful to note that η' and Φ' may be recovered formally from η and Φ through

$$\eta' = \sigma_*\eta, \quad \Phi' = \Phi/\sigma_*^2, \tag{4.3}$$

with σ_* to be evaluated according to (3.15*a*), or (2.10), replacing ϵ with ϵ'. The formulation in the first variable set will be considered first.

4.3. *Reduced equations for finite σ_**

In terms of x, η, ω and Φ, the governing partial differential equation (2.1) for domains I and II reduces, in the limit $\epsilon \to 0$, to a form familiar in the transonic small-disturbance theory (Guderley 1962; Ashley & Landahl 1965; Cole 1969):

$$-(K+\Phi_x)\,\Phi_{xx}+\eta^{-1}(\eta\Phi_\eta)_\eta+\eta^{-2}\Phi_{\omega\omega}=0,\quad K\equiv(M_\infty^2-1)/(\gamma+1)\,M_\infty^2\,\tau\lambda, \tag{4.4}$$

with a remainder of order ϵ^2 and $\epsilon^2\ln\epsilon\Gamma_*$ (i.e. $\tau\lambda^3$ and $\tau\lambda$). To the same degree of approximation, the local pressure coefficient $C_p \equiv 2(p-p_\infty)/\rho_\infty U^2$ and the local Mach number M may be related to Φ_x as

$$\frac{C_p}{2\tau\lambda}=\Phi_x,\quad \frac{M^2-1}{(\gamma+1)\,M_\infty^2\,\tau\lambda}=K+\Phi_x. \tag{4.5}$$

Obviously (4.4) is elliptic in the subsonic and hyperbolic in the supersonic region. In the far field, it admits a uniform free stream (excluding $x\to\infty$, $\eta=O(1)$):

$$\Phi\to 0 \quad \text{as} \quad x^2+\eta^2\to\infty. \tag{4.6}$$

In approaching the x axis, (4.4) admits an expansion for $\eta \ll 1$, using Cartesian tensor notation:

$$\begin{aligned}\Phi \sim\ & D^{(j)}(x)\,l_j(\omega)\,\eta^{-1}+\epsilon C_{ij}(x)\,[2l_i(\omega)\,l_j(\omega)-\delta_{ij}]\,\eta^{-2}+C_0(x)\ln\eta+C_1(x)\\ &+\frac{\partial}{\partial x}\{\tfrac{1}{8}D_x^{(j)}D_x^{(j)}\ln^2\eta-\tfrac{1}{16}[D_x^{(2)}D_x^{(2)}-D_x^{(3)}D_x^{(3)}]\cos 2\omega-\tfrac{1}{8}D_x^{(2)}D_x^{(3)}\sin 2\omega+\ldots\}+\ldots.\end{aligned} \tag{4.7}$$

i and j take on the alternate values 2 and 3 (referring to the y and z axes, respectively); $l_2(\omega)=\cos\omega$ and $l_3(\omega)=\sin\omega$. Terms not written out in (4.7) are of $O(\eta\ln^2\eta)$; they include a feedback in the upwash and sidewash $E_j(x)\,l_j(\omega)\,\eta$. The ϵ in the second terms is introduced in anticipation of the weak quadrupole arising from the inner expansion (cf. (3.28) and (3.18a)).

4.4. *Matching with inner solution*

Equation (4.7) permits matching with the outer limit of the inner expansion (3.28), which is valid for $\epsilon \ll \eta \ll 1$. With φ_{I} and φ_{II} written more fully for the present purpose, (3.28) becomes

$$\begin{aligned}\phi/(\tau Ub)=\Phi \sim\ & (2\pi)^{-1}S_e'(x)\ln\eta+(2\pi)^{-1}D_0(x)\,\eta^{-1}\sin\omega+\beta_0(x)\\ &+\epsilon(2\pi)^{-1}[\tilde{D}_1(x)\,\eta^{-1}\cos\omega+\sigma_*|\ln\epsilon|^{-\frac{1}{2}}\,m_{32}\eta^{-2}\sin 2\omega]+\Phi_{\mathrm{non}},\end{aligned} \tag{4.8}$$

where

$$\beta_0(x)\equiv b_1(x)+\tfrac{1}{8}\sigma_*^2[\Gamma_*\tilde{b}_2(x)+b_2(x)],$$

$$\tilde{D}_1(x)\equiv-\frac{d}{dx}\left\{\bar{y}S_c(x)+(8\pi)^{-1}\sigma_*^2\left[(1-\tfrac{1}{2}|\ln\epsilon|)\,F'm_{32}'-\int_{a_1}^{a_2}y\left([\varphi_0]\frac{\partial^2}{\partial x^2}Z_0-\chi|\ln\epsilon|^{-1}\right)dy\right]\right\}.$$

Φ_{non} is the nonhomogeneous part of $\epsilon\varphi_{\mathrm{II}}+\epsilon^2\varphi_{\mathrm{III}}+\epsilon^3\varphi_{\mathrm{IV}}$ representing the nonlinear and non-transonic corrections in (3.28):

$$\Phi_{\mathrm{non}} \sim (64\pi^2)^{-1}\sigma_*^2\,|\ln\epsilon|^{-1}\,(F_x^2)_x\,(2\ln^2\eta+\cos 2\omega)+\ldots. \tag{4.8a}$$

Terms not written out in the homogeneous part of (4.8) are of order $\epsilon^2\eta^{-2}$, $\epsilon^2\eta^{-1}$, ϵ^2 and η. Terms not written out in (4.8a) for the nonhomogeneous part are of order $\eta\ln^2\eta$ and ϵ. Matching (4.7) with (4.8) determines the strengths of the source, doublet and quadrupole for the outer solution, C_0, $D^{(j)}$ and C_{ij}:

$$\left.\begin{aligned} C_0(x) &= (2\pi)^{-1}S_e'(x), \quad D^{(2)}(x) = \epsilon(2\pi)^{-1}\tilde{D}_1(x), \\ D^{(3)}(x) &= (2\pi)^{-1}\sigma_*|\ln\epsilon|^{-\frac{1}{2}}F(x) \equiv (2\pi)^{-1}D_0(x), \\ C_{ij} &= 0 \quad \text{except} \quad C_{32} = (2\pi)^{-1}\sigma_*|\ln\epsilon|^{-\frac{1}{2}}m_{32}(x). \end{aligned}\right\} \tag{4.9}$$

With $C_1(x) = \beta(x)$, it gives

$$C_1(x) + \epsilon E_j(x)\, l_j(\omega)\, r + \ldots$$

as an outer boundary condition for the inner problem of $\phi/(\tau Ub)$. Having matched the multipole terms in both solutions, the matching of terms in Φ_{non}, including those not written out in (4.8a), with their counterparts in (4.7) follows automatically. Exceptions are terms in Φ_{non} proportional to $\epsilon^2\ln\epsilon\Gamma_*$ or $\epsilon^2\Gamma$, which can be matched only with the higher-order terms associated with the remainder of the transonic small-perturbation equation (4.4).

The Φ_{non} of (4.8a) arises mainly from nonlinear corrections to the Jones equation but is expressible in terms of the doublet strength. Therefore, the important boundary condition on the axis, i.e. (4.7) or (4.8), is specified completely by the strengths of the line source and line doublet, and to a lesser extent, by the line quadrupole, related to the thickness and lift distributions through (4.9). The regular part of the homogeneous solution $C_1(x) + E_j(x)\, l_j(\omega)\,\eta + \ldots$ signifies the feedback from the nonlinear far field, and remains unknown until the boundary-value problem with (4.6) is solved. With the strengths of the multipoles in (4.7) determined, (4.4), (4.6) and (4.7) or (4.8) are the three basic equations defining the boundary-value problem of the outer region, subject to errors of the order ϵ^2 and $\epsilon^2\ln\epsilon\Gamma_*$.

The existence of solutions to the outer problem so formulated can be tested for small σ_*, for which Φ can be expanded in powers of σ_*. The leading term in this case is the axisymmetric solution familiar in the transonic small-disturbance theory. The part of the correction linear in σ_* has been computed on the basis of a finite-difference approximation for certain combinations of $S_e'(x)$ and $F(x)$, using a line-relaxation method (Cheng & Hafez 1973a).

4.5. *Equations for unbounded* σ_*

The formulation of the outer problem in terms of x, η', ω and Φ' belonging to the second variable set can be carried out in a similar manner. The outer expansion of the inner solution to be matched is, of course, that based on ϵ', (3.30). The three basic equations corresponding to (4.4), (4.6) and (4.8) in the limit $\epsilon' \to 0$ for finite K, Γ_* and σ_*^{-1} (i.e. domains II and III) are as follows:

$$\left.\begin{aligned} -(K' + \Phi_x')\,\Phi_{xx}' + (\eta')^{-1}(\eta'\Phi_{\eta'}')_{\eta'} + (\eta')^{-2}\Phi_{\omega\omega}' &= 0, \\ K' \equiv (M_\infty^2 - 1)/(\gamma+1)^2 M_\infty^4\alpha^2\lambda^4|\ln\epsilon'|, & \end{aligned}\right\} \tag{4.10}$$

$$\Phi' \to 0 \quad \text{as} \quad x^2 + \eta'^2 \to \infty \quad (\eta' \neq O(1)), \tag{4.11}$$

$$\Phi' \sim (2\pi)^{-1}\hat{S}'_e(x)\ln\eta' + (2\pi)^{-1}|\ln\epsilon'|^{-\frac{1}{2}}F(x)(\eta')^{-1}\sin\omega + \hat{\beta}_0(x) + \epsilon'(2\pi)^{-1}[\hat{D}_1(x)(\eta')^{-1}\cos\omega + |\ln\epsilon'|^{-\frac{1}{2}}m_{32}(\eta')^{-2}\sin 2\omega] + \hat{\Phi}_{\text{non}} \quad (\epsilon' \ll \eta \ll 1). \tag{4.12}$$

$\hat{S}'_e, \hat{D}_1, \hat{\beta}_0$ and $\hat{\Phi}_{\text{non}}$ are $S'_e, \tilde{D}_1, \beta_0$ and Φ_{non} after division by $\sigma_*^{\frac{2}{3}}$, with ϵ' replacing ϵ. Equations (4.10)–(4.12) remain valid for all finite σ_*^{-1}, including $\sigma_* \to \infty$, with remainders of the order ϵ'^2 and $\epsilon'^2 \ln \epsilon'\Gamma_*$ (i.e. $\alpha^2\lambda^6$ and $\alpha^2\lambda^4$).

The system of outer equations based on the ϵ expansion and that based on the ϵ' expansion are completely equivalent, unless $\sigma_* = 0$ or $\sigma_*^{-1} = 0$. In fact, one system is transformable to the other via $\ln\epsilon' + \ln(\epsilon'/\epsilon) + \ln\epsilon$.† As long as σ_* is neither identically zero nor infinite, *either* one of the system suffices.

4.6. *Shock, vorticity and far field*

Presumably, the smoothness assumptions on the distributions of thickness and lift and on the planform contour may rule out any shock wave on the wing resulting from a local geometrical irregularity. But shocks *may* appear in the outer flow when a part of it becomes supersonic ($K + \Phi_x > 0$). On account of the scale difference in the two regions, such a shock will approach a plane surface (parallel to surfaces of constant x) in the inner region. The inner solution may admit a shock in the form of a discontinuity in the x derivative of $\beta_0(x)$ and $E_j(x)$ in (4.8), or of $C_1(x)$ and $E_j(x)$ in (4.7). This will affect $\epsilon^2\varphi_{\text{III}}$ and $\epsilon^3\varphi_{\text{IV}}$ of the inner solution (3.15), the determination of which will then require use of the Rankine–Hugoniot relation. But this shock discontinuity will not change the order of terms of order $\epsilon^2\varphi_{\text{III}}$ and $\epsilon^3\varphi_{\text{IV}}$ given in (3.28), nor the corresponding terms in (3.30).

The conservation laws governing a gasdynamic discontinuity $x = x^D(\eta, \omega)$ can be written for the outer region, subject to a relative error of $O(\epsilon^2)$ and $O(\epsilon^2 \ln \epsilon\Gamma_*)$, as

$$[\phi_x]\{\langle K + \phi_x\rangle - ([\phi_n]^2 + \eta^{-2}[\phi_\omega]^2)/[\phi_x]^2\} = 0, \tag{4.13}$$

$$[\phi_x] : [\phi_\eta] : [\phi_\omega] = -1 : \partial x^D/\partial\eta : \partial x^D/\partial\omega. \tag{4.14}$$

[ ] and $\langle\,\rangle$ signify the jump and the arithmetical mean, respectively. We also demand that the pressure must be higher on the downstream side, to conform with the second law. For $[\phi_x] \neq 0$, (4.13) yields the shock polar, and (4.14) says that the velocity vector changes in a direction normal to the shock, tantamount to the requirement of continuity in ϕ, i.e.

$$[\phi] = 0. \tag{4.14a}$$

These are the direct results of applying mass and tangential-momentum conservations across the shock under the small-disturbance approximation. The conservation equations for the normal-momentum and energy fluxes are fulfilled, with relative error $O(\epsilon^2, \epsilon^2 \ln \epsilon\Gamma_*)$, when (4.13) and (4.14) are satisfied.

It is easy to arrange (4.4) and the irrotationality condition into suitable divergence forms, of which the weak solutions admit jumps satisfying (4.13) and

† The conversion would give rise to some additional functions of x involving powers of $\ln(\epsilon'/\epsilon)$ in (4.8) or (4.12), which may however be absorbed into the undetermined function $\beta_0(x)$ or $\hat{\beta}_0(x)$.

(4.14) (Cole 1969; Murman & Cole 1971; Garabedian & Korn 1971). These conservation laws, to be sure, include those for slip surfaces and trailing vortex sheets:

$$[\phi_x] = (\partial x^D/\partial \eta)^{-1} = (\partial x^D/\partial \omega)^{-1} = 0.$$

The vorticity and the entropy rise omitted from the analysis cause a relative error of order

$$(\tau\lambda)^2 \quad \text{or} \quad (\alpha^2\lambda^3 \ln \epsilon')^2, \quad \text{i.e.} \quad \epsilon^4\Gamma_*^2|\ln \epsilon|^2 \quad \text{or} \quad \epsilon'^4|\ln \epsilon'|^2\Gamma_*^2.$$

The error incurred by the irrotational assumption is therefore negligible.

Under the requirement $\Phi \to 0$ of (4.6), (4.4) admits three types of behaviour in the far field, depending on K. One may note that, for the slightly supersonic case $K > 0$, even though $K + \Phi_x$ approaches K in the far field, (4.4) cannot be uniformly linearized, because of the cumulative nonlinear distortion of Mach waves over a long distance, as in sonic-boom theories (Whitham 1956; Seebass 1969; Hayes 1971; Cheng & Hafez 1973*a*).

5. Transonic equivalence rule

It follows from the foregoing formulation that the nonlinear outer flow is determined principally by a line source of strength $S_e'(x)$, a line doublet of strength $D_0(x)$, and, to a much lesser degree, by an additional line doublet and a line quadrupole (cf. (4.1); (4.8)). The formulation succeeds, therefore, in dissociating the body geometrical details from the nonlinear mixed-flow problem.

5.1. *Equivalence rule: correlated outer flows*

Assuming that the three basic equations (4.4), (4.6), (4.7) or (48), along with the jump conditions (4.13) and (4.14), yield a unique solution, flows having the same distributions $S_e'(x)$ and $D_0(x)$, with the same transonic parameter K, are *equivalent*. The nonlinear structure of the outer regions (including the shock and the sonic boundary, as well as the characteristic surfaces), when correlated in the variables x, η, ω and Φ, are the same. In particular, the correlated fields of pressure and Mach number for the *same* $S_e'(x)$ and $D_0(x)$ are given by

$$C_p/(\tau\lambda),\ (M^2-1)/\epsilon^2 = f(x/l, \epsilon r/b, \omega; K), \tag{5.1}$$

where

$$\epsilon = [(\gamma+1)\, M_\infty^2\, \tau\lambda^3]^{\frac{1}{2}}.$$

The equivalence rule so stated does not require the same parameters σ_* and Γ_*, nor the same specific-heat ratio γ and the same wing geometry. It is therefore more powerful than the classical transonic similitude. Corresponding to (5.1) is the inviscid drag rise associated with shock waves in the *outer* flow, D_w, in the form

$$D_w/(\rho_\infty U^2 b^2 \tau^2 M_\infty^2) = f(K). \tag{5.2}$$

This follows from the relation between D_w and the entropy increase behind shocks, not elaborated here; but it may also readily be inferred from the form of pressure drag based on the inner solution. Alternatively, (5.1) and (5.2) can be written in terms of the second variable set, with ϵ and K replaced by ϵ' and K' (cf. (4.2*b*)), and with τ^2 in (5.2) replaced by $\tau^2\sigma_*^4$. This is appropriate for $\sigma_* \gg 1$.

In either system, the order of the relative error in the correlations is no greater than the larger of ϵ and ϵ'.

From the two alternative scales of D_w, it is possible to infer that D_w is comparable with, or greater than, the vortex drag D_v only if $\alpha = O(\tau)$, and that the drag associated with the shock loss in the inner region is far less than the greater of D_w and D_v. For a high-aspect-ratio wing with a moderate sweep, the locally supercritical component flow (unaccounted for here) may support spanwise-running shocks and a drag far greater than D_w, depending on the ratio α/τ. The present theory may still be applied to the control of D_w in this case, but is useful only if the local component flow remains shock-free.

5.2. *Lift contribution to the line source*

The application of the equivalence rule requires the same axial lift distribution $F(x)$ or $F'(x)$; but, to keep $S_e'(x)$ invariant, the cross-sectional area $S_c(x)$ cannot remain the same, unless the lift parameter σ_*^2 is negligibly small. The expression for the equivalent line source in (4.1) can be arranged into a more interesting form:

$$S_e'(x) = \frac{d}{dx} S_e(x)$$

$$= \frac{d}{dx}\{S_c(x) + \sigma_*^2[(8\pi)^{-1}(1+\tfrac{1}{2}|\ln\epsilon|^{-1})F_x^2 + \tfrac{1}{2}|\ln\epsilon|^{-1}T(x) + \tfrac{1}{8}\Gamma_* E(x)]\}, \quad (5.3)$$

with
$$T(x) \equiv \frac{1}{4\pi}\int_{a_1}^{a_2}\int_{a_1}^{a_2} [\varphi_0(x,s)]_x [\varphi_0(x,y)]_x \ln\frac{1}{|y-s|}\,ds\,dy, \quad (5.4)$$

$$E(x) = \frac{1}{4\pi}\int_{a_1}^{a_2}\int_{a_1}^{a_2} [\varphi_0(x,s)]_s [\varphi_0(x,y)]_y \ln\frac{1}{|y-s|}\,ds\,dy$$

$$= -\frac{1}{2}\int_{a_1}^{a_2} [\varphi_0]\langle\partial\varphi_0/\partial z\rangle\,dy. \quad (5.5)$$

The following identities have been used:

$$\int_{a_1}^{a_2} [\varphi_0(x,y)]_x\, y\, dy\, \text{p.v.}\int_{a_1}^{a_2} \frac{[\varphi_0(x,s)]_x}{s-y}\,ds = -\frac{1}{2}\left[\int_{a_1}^{a_2}[\varphi_0]_x\,dy\right]^2, \quad (5.6a)$$

$$\int_{a_1}^{a_2}\int_{a_1}^{a_2} [\varphi_0]_{sx}[\varphi_0]_y \ln|y-s|\,ds\,dy = \frac{1}{2}\frac{d}{dx}\int_{a_1}^{a_2}\int_{a_1}^{a_2}[\varphi_0]_s[\varphi_0]_y\ln|y-s|\,ds\,dy. \quad (5.6b)$$

These are valid for the leading-edge behaviour of $[\varphi_0]_x$ assumed (cf. §2.1). (Equation (5.6*a*) was called to our attention by E. Ehlers, A. Chang and P. E. Rubbert of the Boeing Company.)

With (5.5), $E(x)$ signifies the *kinetic energy* in the cross-flow plane; and it gives a non-vanishing cross-sectional area to the equivalent body at $x \to \infty$, identifiable with that of (2.7), §2.2. Interestingly, $T(x)$ assumes the same form as $E(x)$, with the differential pressure $[\varphi_0(x,y)]_x$ replacing the differential side wash $[\varphi_0(x,y)]_y$. It can be shown, via a Fourier representation of $[\varphi]_x$, that $T(x)$ is non-negative, as is $E(x)$. Therefore, (5.3) yields the inequality

$$S_e(x) \geqslant S_c(x). \quad (5.7)$$

That is, the equivalent body corresponding to $S_e'(x)$ has a cross-sectional area greater than that of the geometrical cross-section. Obviously, a greater reduction in body cross-sectional area is needed to compensate for the increase due to lift, if an outer flow corresponding to a chosen $S_e'(x)$ and $D_0(x)$ is to be maintained. But, for a given axial lift distribution $F(x)$ under fixed σ_* and Γ_*, there exists a least value for $(S_e - S_c)$ at each x. With regard to aircraft design application, it is observed that the range of $\sigma_*^{\frac{2}{3}}$ may extend to order unity (Cheng & Hafez 1973*a*, table 1), and that significant nonlinear lift effects may be found even with rather small $\sigma_*^{\frac{2}{3}}$. This is so because the axial length scale characterizing the lift distribution is considerably shorter than that of the whole craft, which makes $d(S_e - S_c)/dx$ much larger.

For wings with bilateral symmetry, those terms proportional to ϵ and ϵ' in (4.8) and (4.12) vanish identically, and the equivalence rule, in this case, becomes as accurate as the partial differential equation itself.

5.3. *Flow symmetry, wall interference and sonic boom*

Since the lift contributes to both $S_e'(x)$ and $D_0(x)$, the line doublet alone can never dominate the nonlinear field, even for a wing without thickness ($\sigma_* \to \infty$). In fact, according to (4.7), (4.9) and (4.12), the equivalent line source is asymptotically stronger than the line doublet as well as Φ_{non}, by a factor of $|\ln \epsilon|^{\frac{1}{2}}$ or $|\ln \epsilon'|^{\frac{1}{2}}$, for any σ_*. The outer flow structure would therefore approach axisymmetry in the strict asymptotic limit $|\ln \epsilon'|^{-\frac{1}{2}} \to 0$, even with $\sigma_* \to \infty$.

It is apparent from above that the smaller of $b\epsilon^{-1}$ and $b(\epsilon')^{-1}$ is the important transverse length scale for transonic wind-tunnel analyses. The distances between walls must be far greater than this scale to treat the wall effect as a (weak) correction. When wall distances are comparable with this scale, a full nonlinear analysis of (4.4) and (4.8), or (4.10) and (4.12), with an appropriate outer boundary condition, is required. In either case, the relative importance of the lift effects is controlled by σ_*, and the effects can be treated in the context of a line doublet and a line source. Similar comment applies to sonic-boom analyses at low supersonic speed. In that case, the nonlinear lift effect through $S_e'(x)$ may contribute a signature no weaker than that of the supersonic doublet in the linear theory (Seebass 1969; Hayes 1971).

In this respect, one must recall a significant part of the equivalence source, $\frac{1}{8}\sigma_*^{\frac{2}{3}}\Gamma_* E(x)$ in (5.3), unaccounted for by classical transonic small-disturbance theory. Thus analyses generated from the latter framework, either numerically or asymptotically, may not describe correctly the outer nonlinear structure (hence the drag rise, the wall-interference effects and the far-field signatures), at least in the domains considered here.†

6. Conclusion

The transonic flow around a planar wing of which the leading-edge sweep angles are not small has been analysed. The flow has an inner region described by the Jones (1946) theory in the leading order, and a larger nonlinear outer

† For three-dimensional flow-field computations based on the transonic small-disturbance theory, see Bailey & Steger (1973), Newman & Klunker (1972).

region determined principally by a line source and a line doublet. The ratio of the transverse length scales for the two regions is ϵ or ϵ' (cf. § 2.1), whichever is greater. The analysis has been carried out as an asymptotic theory in the limit $\epsilon \to 0$ (and $\epsilon' \to 0$) for a finite K (transonic range) and a finite Γ_* (non-slender wing), but an *unrestricted* lift-control parameter $\sigma_* \propto \alpha\tau^{-\frac{1}{2}}\lambda^{\frac{3}{2}}$. The theory confirms the aforementioned structure, and determines explicitly the strengths of the line doublet and line source $D_0(x)$ and $S_e'(x)$ (cf. (5.3)). The equivalence rule established assures that correlated outer flows with the same $D_0(x)$ and $S_e'(x)$ will be essentially the same. The result on $S_e'(x)$ confirms that the equivalent body corresponding to $S_e'(x)$ has an increased cross-sectional area and a non-vanishing afterbody effect, depending on the vortex drag.†

The discussions in § 5 made evident the importance of the lift control in the maintenance of a shock-free supercritical outer flow. It was brought out that, where σ_* is not small, methods of analysis based on the classical transonic small-disturbance theory, unless implemented properly, are inadequate.

The present theory, based on a small ϵ, could be regarded as a development complementing the classical three-dimensional transonic wing formulation (Miles 1959; Guderley 1962). Cole's (1969) review contains the most complete formulation under a set of requirements equivalent to

$$\epsilon^{-1} = O(1), \quad K = O(1), \quad \sigma_* = O(1) \quad \text{and} \quad \Gamma_* = O(\tau^{\frac{2}{3}}),$$

to which the transonic equivalence rule does not apply. On the other hand, the basic difference between the present work and the analyses of Hayes (1954, 1966), Euvrard (1968) and Spreiter & Stahara (1971) is controlled primarily by

$$\sigma \propto \alpha\tau^{-\frac{1}{2}}\lambda^{\frac{3}{2}}.$$

In the cited works, the assumption $\alpha = O(\tau)$ was implicit, which makes the departure from the area rule relatively unimportant, if not altogether negligible. Similar comment applies to the slender-body theory of Oswatitsch & Keune (1954), for which α, τ and λ are all of the same order, so that σ_1 and $\sigma_1^2\Gamma$ are as small as τ^2 and thus negligible (cf. footnote in § 4.1).

From the viewpoint of singular perturbation methods, the result is interesting in that it is the second-order correction to the inner (Jones) solution that gives rise to the important nonlinear lift control on the outer flow. Of particular interest is a crucial part of the line source contributed by terms that dominate neither the inner nor the outer differential equations. Its importance is a consequence of the rapid decay of the Jones (doublet) solution with respect to r, which is overtaken at large r by the non-decaying, source-like, second-order corrections (established in § 3.3).

The present analysis is limited to a planar wing, with rather severe restrictions on the smoothness in the lift and thickness distributions, as well as in its planform. Our principal result on $S_e'(x)$, (5.3) in particular, is derived on the assumption of a 'shock-free entry' at the leading edge without flow separation, with a corresponding smoothness assumption at the trailing edge. For a wing of extremely high

† The inner boundary condition for the outer solution is highly nonlinear not only on account of the lift dependence in S_e but because of the presence of Φ_{non}; cf. (4.8), (4.8a).

aspect ratio, the present formulation must be supplemented with a region next to the wing; the present inner solution then describes the exterior of a lifting line. The formulation as such permits asymptotic analyses of a high-aspect-ratio wing with moderate sweep, involving a locally supercritical component flow.

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